Sol Set 2 PL 221B

Note Title 2/9/2008

Problem 1 We consider the classical 3-dimensional Coulomb problem with an attractive potential, given by the Hamiltonian,

$$H = \frac{1}{2m}\vec{p}^2 - \frac{e^2}{r} \qquad r = |\vec{r}| \tag{0.1}$$

(a) Derive the classical Hamilton equations of motion for $\vec{r}$ and $\vec{p}$;

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} \qquad \dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}$$

$$\dot{\vec{p}} = -\frac{e^2}{r^2}\hat{r} \qquad \dot{\vec{r}} = \frac{\vec{p}}{m}$$

$$\text{EOM:} \quad m\ddot{\vec{r}} = \dot{\vec{p}} = -\frac{e^2}{r^2}\hat{r}$$

(b) Show that orbital angular momentum $\vec{L} = \vec{r}\times\vec{p}$ is conserved, i.e. $d\vec{L}/dt = 0$ when $\vec{r}$ and $\vec{p}$ obey the Hamilton equations of (a);

$$\dot{\vec{L}} = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}} = \underbrace{\frac{\vec{p}\times\vec{p}}{m}}_{\to 0} + \underbrace{\vec{r}\times\left(-\frac{e^2}{r^2}\right)\hat{r}}_{\to 0} = 0$$

(c) Show that the *Runge-Lenz vector* $\vec{A}$, defined below, is a conserved quantity,

$$\vec{A} = \vec{L}\times\vec{p} + me^2\frac{\vec{r}}{r}$$

(d) Show that the Runge-Lenz vector Poisson commutes with H i.e. $\{\vec{A}, H\} = 0$;

$$\dot{\vec{A}} = \vec{L}\times\dot{\vec{p}} + me^2\frac{\dot{\vec{r}}}{r} - me^2\frac{\vec{r}}{r^2}(\dot{\vec{r}}\cdot\hat{r})$$

$$= -\frac{e^2}{r^2}(\vec{r}\times\vec{p})\times\hat{r} + \frac{e^2}{r}\vec{p} - e^2\frac{\vec{r}}{r^2}(\vec{p}\cdot\hat{r})$$

$$= -\frac{e^2}{r^2}\left[r\vec{p} - \vec{p}\cdot\hat{r}\ \vec{r}\right] + \frac{e^2}{r^2}\left[r\vec{p} - \vec{p}\cdot\hat{r}\ \vec{r}\right] = 0$$

$$[\vec{A}, H] = \frac{\partial\vec{A}}{\partial\vec{r}}\frac{\partial H}{\partial\vec{p}} - \frac{\partial\vec{A}}{\partial\vec{p}}\frac{\partial H}{\partial\vec{r}} = \frac{\partial\vec{A}}{\partial\vec{r}}\frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{A}}{\partial\vec{p}}\frac{\partial\vec{p}}{\partial t}$$

$$= \frac{d\vec{A}}{dt} = 0$$

(e) Prove the following Poisson brackets, $\{L_i, L_j\} = \sum_{k=1}^{3} \varepsilon^{ijk} L^k$, and

$$\{A_i, A_j\} = -2mH\sum_{k=1}^{3}\varepsilon^{ijk}L^k \qquad\qquad \{L_i, A_j\} = \sum_{k=1}^{3}\varepsilon^{ijk}A^k \tag{0.3}$$

I will use [ ] to denote Poisson brackets

Start with: $[x^i, p^j] = \delta^{ij}$

$$[x^i, L^j] = \varepsilon^{jkl} x^k [x^i, p^l] = \varepsilon^{jki} x^k = \varepsilon^{ijk} x^k$$

$$[p^i, L^j] = \varepsilon^{jkl}[p^i, x^k]p^l = -\varepsilon^{jil}p^l = \varepsilon^{ijk}p^k$$

$$[L^1, L^2] = [x^2p^3 - x^3p^2, L^2] = x^2[p^3, L^2] - [x^3, L^2]p^2$$

$$= -x^2p^1 + x^1p^2 = +L^3$$

Above calc is symmetric under cyclic permutations of 123, and so we conclude

$$[L^i, L^j] = \varepsilon^{ijk}L^k$$

We will need: $[p^i, 1/r] = -\frac{\partial\, 1/r}{\partial r_i} = \frac{r_i}{r^3}$

$$[L^1, r^{-1}] = [r_2 p^3 - r_3 p^2, 1/r]$$
$$= r_2 [p^3, 1/r] - r_3 [p^2, 1/r] = \frac{r_2 r_3}{r^3} - \frac{r_3 r_2}{r^3} = 0$$

(r is invariant under rotations as expected)

$$[L^1, A^2] = [L^1, L^3 p^1 - L^1 p^3 + me^2 \frac{r_2}{r}]$$
$$= [L^1, L^3] p^1 - L^1 [L^1, p^3] + \frac{me^2}{r} [L^1, r_2]$$
$$= -L^2 p^1 + L^1 p^2 + \frac{me^2}{r} r_3 = A^3$$

Again, invariance under cyclic perms implies
$[L^i, A^j] = \varepsilon^{ijk} A^k$

Now for the difficult piece:

$$[A_i, A_j] = \varepsilon^{imn} \varepsilon^{jkl} [L_m P_n, L_k P_l] \quad ①$$
$$+ \varepsilon^{imn} \left[L_m P_n, me^2 \frac{r_j}{r} \right] \quad ②$$
$$- \varepsilon^{jkl} \left[L_k P_l, me^2 \frac{r_i}{r} \right] \quad ③$$

① $$[L_m p_n, L_k p_l] = L_m [p_n, L_k p_l] - [L_k p_l, L_m] p_n$$
$$= L_m [p_n, L_k] p_l - L_k [p_l, L_m] p_n + [L_m, L_k] p_l p_n$$
$$= \varepsilon^{nkr} p_r p_l L_m - \varepsilon^{lms} L_k p_s p_n + \varepsilon_{mks} L_s p_l p_n$$

$$\varepsilon^{imn}\varepsilon^{jkl}\varepsilon^{nkr} = \varepsilon^{imn}(\delta^{jn}\delta^{lr} - \delta^{jr}\delta^{ln}) = -\varepsilon^{ijm}\delta^{lr} - \varepsilon^{iml}\delta^{jr}$$
$$\varepsilon^{imn}\varepsilon^{jkl}\varepsilon^{lms} = \varepsilon^{imn}(\delta^{jm}\delta^{ks} - \delta^{js}\delta^{nk}) = +\varepsilon^{ijn}\delta^{ks} - \varepsilon^{ikn}\delta^{js}$$
$$\varepsilon^{imn}\varepsilon^{jkl}\varepsilon^{mks} = \varepsilon^{jkl}(-\delta^{ik}\delta^{ns} + \delta^{is}\delta^{kn}) = +\varepsilon^{ijl}\delta^{ns} + \varepsilon^{jnl}\delta^{is}$$

$$\varepsilon^{imn}\varepsilon^{jkl}[L_m p_n, L_k p_l] = -p^2\varepsilon^{ijm}L_m - \varepsilon^{iml}\cancel{p_j} p_l L_m$$
$$- \varepsilon^{ijn}\cancel{L_s p_s} p_n + \varepsilon^{ikn}\cancel{L_k p_j p_n}$$
$$+ \varepsilon^{ijl}\cancel{L_n} p_n p_l$$
$$= -p^2\varepsilon^{ijm}L_m$$

② $$\varepsilon^{imn}\left[L_m p_n, me^2 \frac{r_j}{r}\right]:$$

$$\left[L_m p_n, \frac{r_j}{r}\right] = L_m\left[p_n, \frac{r_j}{r}\right] - \left[\frac{r_j}{r}, L_m\right]p_n$$

$$= L_m [p_n, r_j]\, 1/r - L_m r_j [1/r, p_n]$$
$$- \frac{1}{r}[r_j, L_m] p_n$$

$$= L_m(-\delta_{jn})1/r + L_m \frac{r_j r_n}{r^3} - \frac{1}{r}\varepsilon^{jmk} r_k p_n$$

(2) $\varepsilon^{imn}[L_m p_n, me^2 \frac{r_j}{r}] = me^2\left(\varepsilon^{ijm} L_m \frac{1}{r} + \varepsilon^{imn} \frac{L_m r_j r_n}{r^3} + \frac{1}{r} r_i p_j\right)$

$$\left(\begin{aligned}\varepsilon^{imn} L_m r_n &= \varepsilon^{imn}\varepsilon^{mrs} r_r p_s r_n \\ &= (-\delta^{ir}\delta^{ns} + \delta^{is}\delta^{rn}) r_r p_s r_n \\ &= -r_i p_n r_n + r_n p_i r_n\end{aligned}\right)$$

$\Rightarrow$ The last two terms in (2) are symmetric in i and j, and so cancel the corresponding terms from (3)

$\Rightarrow$ (2) + (3) $= 2me^2 \varepsilon^{ijm} L_m \frac{1}{r}$

Putting everything togeather, we have:

$$[A_i, A_j] = \left(-p^2 + \frac{2me^2}{r}\right)\varepsilon^{ijm} L_m = -2mH\, \varepsilon^{ijm} L_m$$

Problem 2 We now consider the above Coulomb problem in quantum mechanics, and promote H, $\vec{r}$, $\vec{p}$ and $\vec{L}$ to self-adjoint operators using the correspondence principle. The Runge-Lenz vector, as defined above, would not yield a self-adjoint operator; instead, we define its quantum version as the following self-adjoint operator $\vec{A}$,

$$\vec{A} = \frac{1}{2}\vec{L} \times \vec{p} - \frac{1}{2}\vec{p} \times \vec{L} + me^2\frac{\vec{r}}{r} \tag{0.4}$$

(a) Show that the Poisson brackets of (0.3) simply become commutators upon including suitable factors of $i\hbar$, as is familiar from the correspondence principle;

Start with: $[x^i, p^j] = i\hbar\,\delta^{ij}$

$[x^i, L^j] = \varepsilon^{jkl} x^k [x^i, p^l] = i\hbar\,\varepsilon^{jki} x^k = i\hbar\,\varepsilon^{ijk} x^k$

$[p^i, L^j] = \varepsilon^{jkl} [p^i, x^k] p^l = -i\hbar\,\varepsilon^{jil} p^l = i\hbar\,\varepsilon^{ijk} p^k$

$[L^1, L^2] = [x^2 p^3 - x^3 p^2, L^2] = x^2 [p^3, L^2] - [x^3, L^2] p^2$

$= i\hbar(-x^2 p^1 + x^1 p^2) = i\hbar L^3$

Above calc is symmetric under cyclic permutations of 123, and so we conclude

$[L^i, L^j] = i\hbar\,\varepsilon^{ijk} L^k$

We will need: $[p^i, 1/r] = -i\hbar \dfrac{\partial\, 1/r}{\partial r_i} = i\hbar \dfrac{r_i}{r^3}$

$[L^1, r^{-1}] = [r_2 p^3 - r_3 p^2, 1/r]$

$= r_2 [p^3, 1/r] - r_3 [p^2, 1/r] = \left(\dfrac{r_2 r_3}{r^3} - \dfrac{r_3 r_2}{r^3}\right) i\hbar = 0$

(r is invariant under rotations as expected)

$$[L^1, A^2] = [L^1,\ L^3 p^1 - L^1 p^3 + me^2 \frac{r_2}{r}]$$

$$= [L^1, L^3] p^1 - L^1 [L^1, p^3] + \frac{me^2}{r} [L^1, r_2]$$

$$= i\hbar \left(- L^2 p^1 + L^1 p^2 + \frac{me^2}{r} r_3 \right) = i\hbar A^3$$

Again, invariance under cyclic perms implies

$$[L^i, A^j] = i\hbar\, \varepsilon^{ijk} A^k$$

For the last commutator, everything goes through the same as before, except one has to keep track of the ordering (which I did until the very end in problem 1.)

Since: $\vec{A} \sim \frac{1}{2}(\vec{p} \times \vec{L}) - \frac{1}{2}(\vec{L} \times \vec{p}) \sim \frac{1}{2} \varepsilon^{ijk} \left(p^j L^k + L^k p^j \right)$

The additional term guarantees that all the terms will appear with all possible orderings of the operators, and so one obtains

$$[A_i, A_j] = -i\hbar\, 2m\, E_n\, \varepsilon^{ijm} L_m$$

(b) Show that, at a given energy level $E = -\kappa^2/2m$, with $\kappa > 0$, the following combinations

$$\vec{K}_\pm = \frac{1}{2}\vec{L} \pm \frac{1}{2\kappa}\vec{A} \qquad (0.5)$$

obey the commutation relations,

$$[K_+^i, K_-^j] = 0 \qquad [K_\pm^i, K_\pm^j] = i\hbar \sum_k \varepsilon^{ijk} K_\pm^k \qquad (0.6)$$

$$[K_+^i, K_-^j] = \frac{1}{4}[L^i, L^j] - \frac{1}{4\kappa}[L^i, A^j] + \frac{1}{4\kappa}[A^i, L^j] - \frac{1}{4\kappa^2}[A^i, A^j]$$

$$= i\hbar\left(\frac{1}{4}\varepsilon^{ijk}L^k - \frac{1}{4\kappa}\varepsilon^{ijk}A^k - \frac{1}{4\kappa}\varepsilon^{ijk}A^k - \frac{1}{4\kappa^2}(-2m)H\varepsilon^{ijk}L^k\right)$$

Acting now with an arbitrary energy eigenstate from the left (meaning $H = E$), we find

$$[K_+^i, K_-^j] = i\hbar\left(\frac{1}{4}\varepsilon^{ijk}L^k - \frac{1}{4}\varepsilon^{ijk}L^k\right) = 0$$

$$[K_\pm^i, K_\pm^j] = \frac{1}{4}[L^i, L^j] \pm \frac{1}{4\kappa}[L^i, A^j] \pm \frac{1}{4\kappa}[A^i, L^j] + \frac{1}{4\kappa^2}[A^i, A^j]$$

$$= \frac{i\hbar}{4}\varepsilon^{ijk}\left(L^k \pm \frac{A^k}{\kappa} \pm \frac{A^k}{\kappa} + \frac{1}{\kappa^2}(-2m)H L^k\right)$$

Again act on an energy eigenstate:

$$[K_\pm^i, K_\pm^j] = i\hbar\varepsilon^{ijk}\left(\frac{L^k}{2} \pm \frac{A^k}{2\kappa}\right) = i\hbar\varepsilon^{ijk}K_\pm^k$$

(c) Show that $\vec{A}^2 = -\kappa^2(\vec{L}^2 + \hbar^2) + m^2e^4$;

① $\left(\text{Use: } (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = \vec{A}^2\,\vec{B}^2 + (\vec{A}\cdot\vec{B})^2 \right)$ $\quad \left(\begin{array}{l} [L^i, p\cdot L] = 0 \\ \& \ L^i p^i = 0 \end{array} \right)$

$$(\vec{L}\times\vec{p})\cdot(\vec{L}\times\vec{p}) = L^i p^j L^i p^j - L^i \cancel{p^j L^j} p^i \to 0$$
$$= L^i p^j p^j L^i + L^i p^j [L^i, p^j]$$
$$= p^2 L^2 - i\hbar \varepsilon^{ijk} L^i p^j p^k \quad = p^2 L^2$$

$$-(\vec{L}\times\vec{p})\cdot(\vec{p}\times\vec{L}) = L^i p^j p^j L^i - L^i p^j p^i L^j = p^2 L^2$$

$$-(\vec{p}\times\vec{L})\cdot(\vec{L}\times\vec{p}) = p^i L^j L^j p^i - p^i L^j L^i p^j = p^2 L^2 + 4\hbar^2 p^2$$

$$p^j [L^2, p^j] = p^j L^i [L^i, p^j] + p^j [L^i, p^j] L^i = \left(\varepsilon^{ijk} p^j L^i p^k + \varepsilon^{ijk} \cancel{p^j p^k L^i} \right)(-i\hbar) \to 0$$
$$= -i\hbar \varepsilon^{ijk} p^j L^i p^k - i\hbar \varepsilon^{ijk} [p^j, L^i] p^k = \hbar^2 \varepsilon^{ijk} \varepsilon^{jil} p^l p^k$$
$$= 2\hbar^2 p^2$$

$$p^i L^j L^i p^j = p^i L^j \cancel{p^j} L^i + p^i L^j \varepsilon^{ijk} p^k (-i\hbar)$$
0
$$= p^i p^k \cancel{L^j} \varepsilon^{ijk} (-i\hbar) - \hbar^2 \varepsilon^{ijk} \varepsilon^{jkm} p^i p^m$$
0
$$= -2\hbar^2 p^2$$

$$(\vec{p}\times\vec{L})\cdot(\vec{p}\times\vec{L}) = p^i L^j p^i L^j - p^i \cancel{L^j p^j} L^i \to 0$$
$$= L^j p^2 L^j + [p^i, L^j] \cancel{p^i} L^j = p^2 L^2$$
0

$$\left[\tfrac{1}{2}(\vec{L}\times\vec{p}) - \tfrac{1}{2}(\vec{p}\times\vec{L}) \right]^2 = p^2 L^2 + \hbar^2 p^2$$

② $\left(\vec{L}\times\vec{p}\right)\cdot\dfrac{\vec{r}}{r} = \varepsilon^{ijk} L_i p_j \dfrac{r_k}{r} = -\left(L^i \cdot L^i\right) \,{}^1\!/r$

$\dfrac{\vec{r}}{r}\cdot\left(\vec{L}\times\vec{p}\right) = \dfrac{1}{r}\varepsilon^{ijk} r_i L_j p_k = \dfrac{1}{r}\varepsilon^{ijk} r_i p_k L_j$
$+\dfrac{1}{r}\varepsilon^{ijk}\varepsilon^{jkl} r_i p_l \;(i\hbar)$

$= -\dfrac{1}{r}\left(L^i\cdot L^i\right) + \dfrac{i\hbar}{r} 2 r_i p_i$

$-\left(\vec{p}\times\vec{L}\right)\cdot\dfrac{\vec{r}}{r} = -\varepsilon^{ijk} p_i L_j \dfrac{r_k}{r} = -\varepsilon^{ijk} p_i r_k L_j \,{}^1\!/r$
$-\varepsilon^{ijk}\varepsilon^{jkm} p_i r_m \,{}^1\!/r\, (i\hbar)$

$= -\left(L^i \cdot L^i\right){}^1\!/r - i\hbar\, 2\left(p_i\, r_i\right){}^1\!/r$
$= -L^2/r + \hbar^2 2\left(\dfrac{r^2}{r^3}\right) - \dfrac{i\hbar}{r} 2\left(p_i\, r_i\right)$

$-\dfrac{\vec{r}}{r}\cdot\left(\vec{p}\times\vec{L}\right) = -\varepsilon^{ijk}\dfrac{r_i}{r} p_j L_k = -\dfrac{1}{r}L^2$

Combining gives the cross terms:

$$-\frac{2me^2}{r}L^2 + \frac{1}{2}\left(\frac{2me^2\hbar^2}{r} + \frac{i\hbar 2}{r}\underbrace{[r_i, p_i]}_{=i\hbar 3} me^2\right)$$

$$= -\frac{2me^2 L^2}{r} - \frac{2me^2\hbar^2}{r}$$

Combining everything now gives:

$$\vec{A}^2 = \left(p^2 - \frac{2me^2}{r}\right)\left(L^2 + \hbar^2\right) + m^2e^4$$

$$= 2mH\,(L^2 + \hbar^2) + m^2e^4 = -\hbar^2(L^2+\hbar^2) + m^2e^4$$

(d) Deduce therefrom the expression for the energy E in terms of $(\vec{K}_+)^2$ and $(\vec{K}_-)^2$;

$$\vec{L}\cdot\vec{A} + \vec{A}\cdot\vec{L} = -\frac{1}{2}\vec{L}\cdot\cancel{(\vec{p}\times\vec{L})} + me^2\vec{L}\cdot\frac{\vec{r}}{r} + \frac{1}{2}\cancel{(\vec{L}\times\vec{p})}\cdot\vec{L} + me^2\frac{\vec{r}}{r}\cdot\vec{L}$$

$$\vec{L}\cdot\frac{\vec{r}}{r} + \frac{\vec{r}}{r}\cdot\vec{L} = \varepsilon^{ijk} r_i p_j r_k r^{-1} + r^{-1}\varepsilon^{ijk} r_i p_j r_k = 0$$

$$K_\pm^2 = \frac{1}{4}\vec{L}^2 \pm \frac{1}{4\hbar}\left(\vec{L}\cdot\vec{A} + \vec{A}\cdot\vec{L}\right) + \frac{1}{4\hbar^2}\vec{A}^2$$

$$= \frac{1}{4}\vec{L}^2 - \frac{1}{4}L^2 - \frac{\hbar^2}{4} + \frac{m^2e^4}{4\hbar^2} = \frac{m^2e^4}{4\hbar^2} - \frac{\hbar^2}{4}$$

$$\Rightarrow \boxed{K_+^2 = K_-^2}$$

$$E = -\frac{\hbar^2}{2m} = m^2e^4\left(\hbar^2 + 4K_\pm^2\right)^{-1}$$

(e) Use this result, and the representation theory for the "angular momentum algebras" of $\vec{K}_+$ and $\vec{K}_-$ to derive the full bound state spectrum for the Coulomb problem.

We showed $\vec{K}_+$ and $\vec{K}_-$ form representations of $SU(2)$ and $K_+^2 = K_-^2$

$$\Rightarrow \quad K_+^2 = K_-^2 = \hbar^2 j(j+1) \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$\Rightarrow \quad E = \frac{m^2 e^4}{\hbar^2 2m(1+4j^2+4j)}$$

Define: $n = 2j+1 \Rightarrow j = \frac{n-1}{2}$

$$4j^2 + 4j + 1 = n^2 - 2n + 1 + 2n - 2 + 1 = n^2$$

$$\Rightarrow E = \frac{m e^4}{2\hbar^2 n^2} \qquad n = 1, 2, 3, \ldots$$

This is the standard formula for the energy levels of the Hydrogen atom. See (A.6.3) in Sakurai.